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Factorisation of the sum of two non zero distinct squares into the product of two sums of two non zero squares : one of them is a prime number

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Abstract : This work has three parts :

1/ the Theorem whith proof calls upon two important results :

a/ Every sum of two non zero distinct squares is divisible by an odd prime number that is the sum of two coprime squares.

[Habib Lebsir's theorem]

b/ Every divisor of the sum of two coprime squares is itself the sum of two non zero squares. [Fermat's theorem]

2/ the construction of infinite sequences of sums of two non zero distinct squares factorisable into the product of two sums of two non zero squares : one of them is an odd prime number.

3/ the List of the first twenty prime numbers sums of two squares :

Keywords: Factorisation, sum of two squares, distinct, non zero, prime, product, coprime, composite.

Introduction

In a previous publication, I proved the existence of an odd prime divisor of the sum of two non zero distinct squares $(a^2 + b^2)$ that is the sum of two coprime squares $(x^2 + y^2)$.

$(a^2 + b^2) = (x^2 + y^2).q$; x and y coprime

Habib Lebsir's Theorem ,Ijr journal , ISSN 2438-6848 , Volum8-ISSue5 ,may 2021 In this publication I focus on the nature of the quotient q of the division of $(a^2 + b^2)$ by $(x^2 + y^2)$ and ask the following question :

what are the conditions on a and b that make

q expressed as the sum of two non zero squares ?

the answer is given in the following pages.

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Preface

- ✓ the present work is a modest comtribution to the progress of mathematics and number theory in particular.
- ✓ It helps the interested readers to better undersland the divisibility in the set of the sums of two non zero distinct squares.
- ✓ I hope that all readers of the theorem will appreciate it.
- ✓ Note that,in the previous publication mentioned in the introduction,the phrase ″prime between themeselves ″ means ″coprime″

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Theorem:

Let a , b be two non zero distinct naturals and d their gcd .
$(a^2 + b^2)$ is factorisable into the product of two non zero squares : one of them is a prime number if :

$$\begin{cases} (a^2 + b^2)/d^2 \text{ is composite (non prime)} \\ \text{or} \\ (a^2 + b^2)/d^2 \text{ is prime and } d^2 \text{ is expressed as the sum of two non zero squares} \end{cases}$$

Proof :

1/ Suppose that $(a^2 + b^2)/d^2$ is composite

a/ Let a and b be coprime (d = 1)

$a^2 + b^2$ is divisible by an odd prime number that is the sum of two coprime squares ($x^2 + y^2$) (as the sum of two non zero distinct squares).

[Habib Lebsir's Theorem] thus : $(a^2 + b^2) = (x^2 + y^2).q$

q is the sum of two non zero squares $(z^2 + t^2)$ [Fermat's theorem]

Therefore : $(a^2 + b^2) = (x^2 + y^2).(z^2 + t^2)$

$x^2 + y^2$: odd prime ; $z^2 + t^2$ sum of two non zero squares

Example : $(7^2 + 3^2) = 58 = 29 \times 2 = (2^2 + 5^2).(1^2 + 1^2)$

b/ a and b are not coprime ($d \neq 1$).

$a = d.a'$, $b = d.b'$ a′ and b′ are coprime.

$(a^2 + b^2) = d^2.(a'^2 + b'^2)$; $(a'^2 + b'^2)$ is composite (non prime)

$(a'^2 + b'^2)$ is divisible by an odd prime number that is the sum of two coprime squares $(x^2 + y^2)$ (as the sum of two non zero distinct squares)

[Habib Lebsir's Theorem].

$(x^2 + y^2)/(a'^2 + b'^2)$ then $(a'^2 + b'^2) = (x^2 + y^2).q'$

q′ is the sum of two non zero squares because $q'/(a'^2 + b'^2)$ and a′ and b′ are coprime.[Fermat's theorem]

Therefore : it exists two non zero naturals u , v such as : $u^2 + v^2 = q'$

so : $(a'^2 + b'^2) = (x^2 + y^2).(u^2 + v^2)$

and $a^2 + b^2 = d^2(a'^2 + b'^2) = d^2(x^2 + y^2).(u^2 + v^2)$

$a^2 + b^2 = (x^2 + y^2)[(du)^2 + (dv)^2] = (x^2 + y^2).(z^2 + t^2)$ with : $z = du$, $t = dv$.

$a^2 + b^2 = (x^2 + y^2).(z^2 + t^2)$; $x^2 + y^2$ is an odd prime number, x and y are coprime

Examples :

•) $6^2 + 8^2 = 100 = 5 \times 20 = (1^2 + 2^2).(2^2 + 4^2)$
$d = 2$, $d^2 = 4$, $5 = 1^2 + 2^2$ prime

••) $3^2 + 9^2 = 90 = 5 \times 18 = (1^2 + 2^2).(3^2 + 3^2)$
$d = 3$, $d^2 = 9$, $5 = 1^2 + 2^2$ odd prime

2/ $(a^2 + b^2)/d^2$ is prime and d^2 is expressed as the sum of two non zero squares :

$(a^2 + b^2)/d^2 = (a'^2 + b'^2)$ then : $a^2 + b^2 = d^2(a'^2 + b'^2) = (a'^2 + b'^2).(x^2 + y^2)$

$(x^2 + y^2)$: sum of two non zero squares

$(a'^2 + b'^2)$: odd prime number ($a' \neq b'$)

Examples :

•) $a = 10$, $b = 15$, $d = 5$ $d^2 = 25$
$a^2 + b^2 = 10^2 + 15^2 = 325 = 25 \times 13$
$d^2 = 25 = (3^2 + 4^2)$, $13 = (2^2 + 3^2)$ prime ; $(10^2 + 15^2) = (2^2 + 3^2).(3^2 + 4^2)$

••) $a = 10$, $b = 20$, $d = 10$, $d^2 = 100$

$a^2 + b^2 = 10^2 + 20^2 = 500 = 100\times 5$

$d^2 = 100 = (6^2 + 8^2)$, $5 = (1^2 + 2^2)$ prime ; $(10^2 + 20^2) = (6^2 + 8^2)(1^2 + 2^2)$

Remark 1 :

We have : $5^2 = (3^2 + 4^2)$ then for every $n \in IN$: $(5n)^2 = (3n)^2 + (4n)^2$

non zero natural numbers of the form $(5n)^2$ are expressed as sums of two non zero squares, we can state :

If : $d = 5n$ ($n \in IN$) and $(a^2 + b^2)/d^2$ is a prime number

Then : $(a^2 + b^2) = [(3n)^2 + (4n)^2][a'^2 + b'^2]$

$(a'^2 + b'^2)$ is an odd prime ($a' \neq b'$)

Consequence :

Construction of infinite sequences of sums of two non zero distinct squares which terms are factorisable into the product of two sums of two non zero squares :one of them is an odd prime number.

let's consider the sequence : $[(3n)^2 + (4n)^2][x^2 + y^2]$,$n \in IN^*$

$x^2 + y^2$ an odd prime number(x and y are coprime)

we have : $[(3n)^2 + (4n)^2] = 25.n^2 = (5n)^2$ then

$$[(3n)^2 + (4n)^2](x^2 + y^2) = (5n)^2.(x^2 + y^2) = (5nx)^2 + (5ny)^2 = (a_n)^2 + (b_n)^2$$

with : $a_n = 5nx$; $b_n = 5ny$

$d_n = 5n$ because $a_n/d_n = x$, $b_n/d_n = y$, and : x and y are coprime

therefore : $(a_n)^2 + (b_n)^2 = (d_n)^2(x^2 + y^2) = [(3n)^2 + (4n)^2].[x^2 + y^2]$

$(d_n)^2 = (3n)^2 + (4n)^2$ is the sum of two non zero squares $(3n)^2$ and $(4n)^2$

$x^2 + y^2$ is an odd prime number

Example :

•) $[(3n)^2 + (4n)^2].[1^2 + 2^2] = (5n)^2 + (10n)^2$; $1^2 + 2^2 = 5$ prime

••) $[(3n)^2 + (4n)^2].[2^2 + 3^2] = (10n)^2 + (15n)^2$; $2^2 + 3^2 = 13$ prime

•••) $[(3n)^2 + (4n)^2].[1^2 + 4^2] = (5n)^2 + (20n)^2$; $1^2 + 4^2 = 17$ prime

Remark 2 :

Only sums of two non zero distinct squares and d^2 is not expressible as the sum of two non zero squares, are not factorisable into the product of two sums of two non zero squares.

Like : $6^2 + 9^2 = 117 = 9\times 13 = 3^2(2^2 + 3^2)$

$d = 3$; $d^2 = 9$ is not the sum of two non zero squares, 13 is a prime number.

In particular: Every odd prime number that is the sum of two coprime squares is not factorisable into the product of two sums of two non zero squares because : if $p = x^2 + y^2$ is a prime number with x and y coprime

then : $x^2 + y^2 = (x^2 + y^2).1^2$ and 1^2 is not expressible as the sum of two non zero squares.

List of the first twenty prime numbers sums of two squares :

01	$2=1^2+1^2$
02	$5=1^2+2^2$
03	$13=2^2+3^2$
04	$17=1^2+4^2$
05	$29=2^2+5^2$
06	$37=1^2+6^2$
07	$41=4^2+5^2$
08	$53=2^2+7^2$
09	$61=5^2+6^2$
10	$73=3^2+8^2$

11	$89=5^2+8^2$
12	$97=4^2+9^2$
13	$101=1^2+10^2$
14	$109=3^2+10^2$
15	$113=7^2+8^2$
16	$137=4^2+11^2$
17	$149=7^2+10^2$
18	$157=6^2+11^2$
19	$173=2^2+13^2$
20	$181=9^2+10^2$